Name of College :→ S.S. College, J-Bad. Page-1 28-07-2020

Depth - Mathematics

TOPIC - Problem on Curvature (Contd)
(Diff. calculus)

Class - B.Sc I (Hons)

Date - 28-07-2020

Time - 10.15 A.M to 11.00 A.M
11.00 A.M to 11.45 A.M.

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Problem :→ If CP and CD be a pair of conjugate semi-diameters of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

prove that radius of curvature at P is $\frac{CD^3}{ab}$

Solution :→ Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $P(a\cos t, b\sin t)$ be a point

then co-ordinate of D is

$(a\cos(\pi/2 + t), b\sin(\pi/2 + t))$

i.e $(-a\sin t, b\cos t)$

for radius of curvature at P

$x = a\cos t \Rightarrow x' = -a\sin t$ $\quad x'' = -a\cos t$

$y = b\sin t \Rightarrow y' = b\cos t$ $\quad y'' = -b\sin t$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab\sin^2 t + ab\cos^2 t} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

Now

$$CD^2 = (0 + a\sin t)^2 + (0 - b\cos t)^2$$
$$= a^2\sin^2 t + b^2\cos^2 t$$
$$CD = (a^2\sin^2 t + b^2\cos^2 t)^{1/2}$$
$$CD^3 = (a^2\sin^2 t + b^2\cos^2 t)^{3/2}$$

thus the radius of curvature at P

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$
$$= \frac{CD^3}{ab}$$

Problem 2. If ρ and ρ' be the radii of curvature at the extremities of two conjugate diameters of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Prove that

$$(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2.$$

Solution :→ By the question

Given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let the co-ordinate of P be $(a\cos t, b\sin t)$ then co-ordinate of D be $(-a\sin t, b\cos t)$ where CP and CD be two semi-conjugate diameters.

Now for radius of curvature at P

$x = a\cos t \Rightarrow x' = -a\sin t$ $\therefore x'^2 = a^2\sin^2 t$

$\Rightarrow x'' = -a\cos t$ $y'^2 = b^2\cos^2 t$

$y = b\sin t \Rightarrow y' = b\cos t$ $x'y'' = ab\sin^2 t$

and $y'' = -b\sin t$ $x''y' = -ab\cos^2 t$

Now ρ = radius of curvature at P

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab\sin^2 t + ab\cos^2 t}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \quad \text{—— (1)}$$

for radius of curvature at D

$x_1 = -a\sin t$ $y_1 = b\cos t$

$x_1' = -a\cos t$ $y_1' = -b\sin t$

$x_1'' = a\sin t$ $y'' = -b\cos t.$

$$\rho' = \frac{(x_1'^2 + y_1'^2)^{3/2}}{x_1'y_1'' - x_1''y_1'}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab\cos^2 t + ab\sin^2 t}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \quad \text{—— (II)}$$

from (1)

$$a^2\sin^2 t + b^2\cos^2 t = (\ell ab)^{2/3}$$ 28-07-

from (1) $a^2\cos^2 t + b^2\sin^2 t = (\ell' ab)^{2/3}$

On Adding

$$a^2 + b^2 = (\ell ab)^{2/3} + (\ell' ab)^{2/3}$$

$$= (ab)^{2/3}\left(\ell^{2/3} + \ell'^{2/3}\right)$$

Prove

Problem 3

Find the radius of curvat at any point of the curve

$$x = a(\theta + \sin\theta) \quad y = a(1-\cos\theta)$$

Solution :→

Since Radius of curvature

$$\ell = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} \quad \text{—— (1)}$$

Here $x = a(\theta + \sin\theta) \Rightarrow x' = a(1+\cos\theta)$

$x'' = -a\sin\theta$

$y = a(1-\cos\theta) \Rightarrow y' = a\sin\theta$

$y'' = a\cos\theta$

$\therefore x'^2 = a^2(1 + 2\cos\theta + \cos^2\theta)$

$y'^2 = a^2\sin^2\theta$

$\therefore x'^2 + y'^2$

$= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$

$= 2a^2(1+\cos\theta)$

$= 2a^2 \times 2\cos^2\theta/2$

$= 4a^2\cos^2\theta/2$

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$= a(1+\cos\theta) \cdot a\cos\theta - (-a\sin\theta)(a\sin\theta)$

$= a^2[\cos\theta + \cos^2\theta + \sin^2\theta]$

$= a^2(1+\cos\theta)$

$= a^2 \times 2\cos^2\theta/2$

Putting all these values in (1)

we have ρ = radius of curvature

$$= \frac{[4a^2\cos^2\theta/2]^{3/2}}{2a^2\cos^2\theta/2}$$

$$= \frac{(2a\cos\theta/2)^3}{2a^2\cos^2\theta/2} = \frac{\cancel{8}^{4}\cancel{a^3}^{a}\cancel{\cos^3\theta/2}^{\cos\theta/2}}{\cancel{2}\cancel{a^2}\cancel{\cos^2\theta/2}}$$

$= 4a\cos\theta/2$

Problem 4 :→ Show that

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

Solution

We know $\sin\psi = \frac{dy}{ds}$ — (I) where ψ is the angle

$\cos\psi = \frac{dx}{ds}$ — (II)

$\tan\psi = \frac{dy}{dx}$ — (III)

$\rho = \frac{ds}{d\psi}$ — (IV)

Diff (I) with respect to s

$\cos\psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2}$

$\cos\psi \frac{1}{\rho} = \frac{d^2y}{ds^2}$ ——— (1)

Diff. with respect to s

$$-\sin\psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2}$$

$$\Rightarrow -\sin\psi \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2}$$

Since $k = \frac{d\psi}{ds}$

$\rho = \frac{1}{k} = \frac{ds}{d\psi}$

S ca

Now $\cos^2\psi \frac{1}{\rho^2} + \sin^2\psi \frac{1}{\rho^2} = \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2$

$$\Rightarrow \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

proved.

Problem 5- If ρ be the radius of curvature of a parabola at a point whose distance along the curve from a fixed point is s, prove that

$$3\rho \frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9 = 0$$

Solution → Say equation of Parabola be

$$y^2 = 4ax$$

So that $y = 2\sqrt{ax}$

$$y_1 = 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$

$$y_2 = \sqrt{a}\left(-\frac{1}{2}\right) x^{-3/2}$$

$$= -\frac{1}{2} \frac{\sqrt{a}}{x^{3/2}}$$

Now $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1+\frac{a}{x}\right]^{3/2}}{-\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}}}$

$$\Rightarrow \rho = -\frac{2(x+9)^{3/2}}{\sqrt{9}} \quad \text{--- (1)}$$

$$-\frac{2(x+9)^{3/2}}{\sqrt{9}}$$ In magnitude

$$\frac{d\rho}{ds} = \frac{d\rho}{dx} \cdot \frac{dx}{ds}$$

Now $\frac{d\rho}{dx} = \frac{2}{\sqrt{9}} \cdot \frac{3}{2} (x+9)^{1/2}$

$$= \frac{3}{\sqrt{9}} \sqrt{x+9}$$

$$\frac{dx}{ds} = \sqrt{1+y_1^2} = \sqrt{1+\frac{9}{x}} = \sqrt{\frac{x+9}{x}}$$

$$\therefore \frac{ds}{dx} = \sqrt{\frac{x}{x+9}}$$

$$\therefore \frac{d\rho}{ds} = \frac{3}{\sqrt{9}} \sqrt{x+9} \cdot \sqrt{\frac{x}{x+9}} = \frac{3\sqrt{x}}{\sqrt{9}}$$

$$\frac{d^2\rho}{ds^2} = \frac{d}{ds}\left(\frac{d\rho}{ds}\right) = \frac{d}{ds}\left(\frac{3\sqrt{x}}{\sqrt{9}}\right) = \frac{d}{dx}\left(\frac{3\sqrt{x}}{\sqrt{9}}\right) \cdot \frac{dx}{ds}$$

$$= \frac{1}{2} \frac{3}{\sqrt{9}\sqrt{x}} \sqrt{\frac{x}{x+9}} = \frac{3}{2\sqrt{9}} \frac{1}{\sqrt{x+9}}$$

Now $3\rho \frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9$

$$= 3 \cdot \frac{2}{\sqrt{9}} (x+a)^{3/2} \frac{3}{2\sqrt{9}} \frac{1}{\sqrt{x+9}} - \frac{9x}{9} - 9$$

$$= \frac{9}{a}(x+9) - \frac{9(x+9)}{9}$$

$$= 0$$

Hence the result.